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1992 J. Phys. A: Math. Gen. 25 L629

(http://iopscience.iop.org/0305-4470/25/11/001)

J. Phys. A: Math. Gen. **25** (1992) L629–L631. Printed in the UK

LETTER TO THE EDITOR

Universal *R*-matrix for a two-parametric quantization of gl(2)

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Received 10 February 1992

Abstract. For the two-parametric quantized group $\mathrm{gl}_{q,s}(2)$, we calculate the explicit form of the universal R-matrix.

The deformations of the algebra $\mathrm{gl}_{q,s}(2)$ with two deformation parameters have been studied in many papers (Kulish 1990, Schirrmacher *et al* 1991, Burdík and Hlavatý 1991, Schirrmacher 1991). The starting point for a construction of this two-parametric deformation is a two-parametric solution of the Yang–Baxter equation (YBE)

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} \tag{1}$$

found by Hlavatý (1987), Demidov *et al* (1990) and Schirrmacher *et al* (1991):

$$R_{q,s}:=\begin{pmatrix} q & 0 & 0 & 0\\ 0 & s & 0 & 0\\ 0 & \Omega & 1/s & 0\\ 0 & 0 & 0 & q\end{pmatrix}q^{-1/2}$$

where

$$\Omega=q-q^{-1}. \tag{2}$$

It was found by Reshetikhin, Takhtajan and Faddeev (1989) that the defining relations for the quantum enveloping algebra can be obtained from the relations

$$\begin{aligned}&R^{+}L_1(\varepsilon)L_2(\varepsilon)=L_2(\varepsilon)L_1(\varepsilon)R^{+}\\ &R^{+}L_1(+)L_2(-)=L_2(-)L_1(+)R^{+}\end{aligned} \tag{3}$$

where

$$R^{+}:=PRP \tag{4}$$

$$P:=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\varepsilon=\pm \qquad L_1(\varepsilon)=L(\varepsilon)\otimes 1 \qquad L_2(\varepsilon)=1\otimes L(\varepsilon). \tag{5}$$

The explicit calculation for the R-matrix (1) was given by Kulish (1990). In this case

$$L(+)=\begin{pmatrix} k_1 & Y \\ 0 & k_2 \end{pmatrix} \qquad L(-)=\begin{pmatrix} l_1 & 0 \\ X & l_2 \end{pmatrix} \qquad \text{and } R=R_{q,s}. \tag{6}$$

The algebra $\mathrm{gl}_{q,s}\equiv \mathrm{U}_{q,s}(\mathrm{gl}(2))$ is obtained as the free algebra generated by $(k_1, k_2, l_1, l_2, X, Y)$ and factorized by the ideal corresponding to the relations

$$\begin{aligned}
&k_1Y=(qs)^{-1}Yk_1 \qquad k_2Y=qs^{-1}Yk_2 \qquad [k_1,k_2]=0\\
&l_1X=(q)^{-1}sXl_1 \qquad l_2X=qsXl_2 \qquad [l_1,l_2]=0\\
&k_1X=qsXk_1 \qquad k_2X=q^{-1}sXk_2 \qquad [k_i,l_j]=0\\
&l_1Y=qs^{-1}Yl_1 \qquad l_2Y=(qs)^{-1}Xl_2\\
&[X,Y]_s\equiv sXY-1/sYX=\Omega(k_2l_1-l_2k_1).
\end{aligned} \tag{7}$$

The familiar commutation relations of the $\mathrm{gl}_{q,s}(2)$

$$\begin{aligned}
&[J_0,J_\pm]=\pm J_\pm \qquad [J_+,J_-]=\frac{\sin(2\eta J_0)}{\sinh\eta}\\
&[Z,J_\pm]=0 \qquad [Z,J_0]=0
\end{aligned} \tag{8}$$

can be obtained from (7) by the substitutions

$$\begin{aligned}
&k_1=(qs)^{-J_0+Z} \qquad k_2=\left(\frac{s}{q}\right)^{-J_0+Z} \qquad Y=\Omega J_+(s)^{-J_0}\\
&l_1=\left(\frac{s}{q}\right)^{-J_0-Z} \qquad l_2=(qs)^{-J_0-Z} \qquad X=-\Omega J_-(s)^{-J_0}.
\end{aligned} \tag{9}$$

We can see that the above commutation relations (8) are not deformed with s and this deformation parameter is only in the comultiplication for this algebra

$$\begin{aligned}
&\Delta(J_+)=q^{-J_0}(qs)^{Z}\otimes J_+ + J_+\otimes q^{J_0}\left(\frac{s}{q}\right)^{Z}\\
&\Delta(J_-)=q^{-J_0}(qs)^{-Z}\otimes J_- + J_-\otimes q^{J_0}\left(\frac{s}{q}\right)^{-Z}\\
&\Delta(J_0)=J_0\otimes 1+1\otimes J_0,\ \Delta(Z)=Z\otimes 1+1\otimes Z
\end{aligned} \tag{10}$$

and in the antipod

$$S(Z)=-Z \qquad S(J_0)=-J_0 \qquad S(J_\pm)=-q^{\pm 1}s^{\mp 2Z}J_\pm. \tag{11}$$

For a co-unit we have again

$$\varepsilon(Z)=\varepsilon(J_0)=\varepsilon(J_\pm)=0. \tag{12}$$

Theorem 1. The algebra $\mathrm{gl}_{q,s}(2)$ with Δ, ε and S given by (10), (12) and (11) is a Hopf algebra.

Proof. By direct verification of the axioms of the Hopf algebra.

A Hopf algebra is a quasitriangular Hopf algebra (see Drinfeld 1986) if the comultiplications Δ, Δ' are related by conjugation

$$\Delta'(a) \equiv \sigma \qquad \Delta(a) = R\Delta(a)R^{-1} \qquad R \in A \otimes A \qquad \sigma(a \otimes b) = b \otimes a \tag{13}$$

for any $a \in A$, and the following conditions are satisfied

$$\begin{aligned} (id \otimes \Delta)(R) &= R_{13}R_{12} \\ (\Delta \otimes id)(R) &= R_{13}R_{23} \\ (S \otimes id)(R) &= R^{-1} \end{aligned} \tag{14}$$

For the $\mathrm{sl}_q(2)$ the universal element

$$R = q^{(J_0 \otimes J_0)/2} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} (q^{J_0/2}J_+ \otimes q^{-J_0/2}J_-)^n q^{n(n-1)/2} \tag{15}$$

was given by Kirillov and Reshetikhin (1988) and one can check it satisfies (13) and (14).

An explicit form of universal element is given in the following theorem.

Theorem 2. The algebra $\mathrm{gl}_{q,s}(2)$ is the quasitriangular Hopf algebra with the universal element $R_{q,s} \in \mathrm{gl}_{q,s}(2) \otimes \mathrm{gl}_{q,s}(2)$ which is given by the formula

$$\begin{aligned} R_{q,s} = q^{2(J_0 \otimes J_0 + J_0 \otimes Z - Z \otimes J_0)} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{n(n-1)/2} (q^{J_0}(qs)^{-Z} J_+)^n \\ \otimes \left(q^{-J_0} \left(\frac{s}{q} \right)^Z J_- \right)^n . \end{aligned} \tag{16}$$

Proof. By direct verification of the relation in (13) and (14).

In this letter we have presented the universal element R of the quantum group $\mathrm{gl}_{q,s}(2)$. We propose that this explicit realization could play a role in nonlinear integrable systems in physics.

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